Numerical Wonders: Notes

I. Square Wonders

Trailing 5's squaring trick: to find the square of a two-digit number ending in 5, take the tens digit, add one, and multiply this by the original tens digit. Then append 25 to the end of the result.

Example: to find 45^2, take the tens digit, 4. Add one to get 5. Multiply it by the original tens digit, 4, to get 5*4 = 20. Add 25 at the end: 2025.

Note: This also works for numbers with three digits or more, but now you take everything to the left of the ones digit. For 135^2, take 13, add 1 to get 14, multiply by 13 to find 182. Add 25: 18225.

1's squaring pattern: If you square a number with *n* 1's, you get a number such that the first *n* digits increase from 1 to *n*, and the last *n* digits decrease from *n* to 1. Note that *n* has to be between 1 and 9.

Example: 11111^2 = 123454321 (in this case, *n* = 5)

40's squaring pattern: As you go from 41^2 to 50^2, the number formed by the left two digits increases by one from 16 to 25, while the number formed by the last two digits go from 9^2 down to 0^2.

Observe:	41^2 = 1681	First two digits = 16, last two digits are 9^2 = 81
	42^2 = 1764	First two digits = 17, last two digits are 8^2 = 64
	43^2 = 1849	First two digits = 18, last two digits are 7^2 = 49
	…	
	49^2 = 2401	First two digits = 24, last two digits are 1^2 = 01
	50^2 = 2500	First two digits = 25, last two digits are 0^2 = 00

II. Fractional Wonders

Pattern of ninths: 1/9 = 0.1111…, 2/9 = 0.2222…, 3/9 = 1/3 = 0.3333…

Pattern of 11ths: Take numerator and multiply by 9. This is what repeats.

Example: 1/11 = 0.090909…, 5/11 = 0.454545… (5*9 = 45), 7/11 = 0.636363… (7*9 = 63)

Rotating sevenths: The repeating parts of fractions 1/7, 2/7, …, 6/7 are just rotations of each other.

Observe:	$1/7 = 0.\overline{142857}$	A line over the digits means that it repeats.
	$2/7 = 0.\overline{285714}$	It's just 1/7 but rotated two digits to the left!
	$3/7 = 0.\overline{428571}$	It's just 2/7 but rotated one digit to the right!

Note: With these tricks, you can also find fractions like 22/7 and 71/9 easily: simply break 22/7 into $22/7 = 3 + 1/7 = 3.\overline{142857}$ and similarly, $71/9 = 7 + 8/9 = 7.\overline{8}$.

The amazing repeating decimal: $1/9^2 = 1/81 = 0.\overline{012345679}$, every digit in order except 8

$1/99^2 = 1/9801 = 0.\overline{0001020304 \ldots 95969799}$, goes 00, 01, 02, …, 99 in order, except 98.

Note that 98 is the first two digits of 9801.

Similarly, $1/999^2 = 1/998001 = 0.\overline{000001002 \ldots 996997999}$ goes 000, 001, 002, …, 999, in order, except 998. Note that 998 is the first three digits of 998001.

III. Coincidental Wonders

e: Look at the digits of Euler's number: 2.7 1828 1828 459045 … (repeat 1828 twice, then 45-90-45 triangle). This is purely a coincidence – there is no mathematical reason.

Number of the beast: The number 666 is interesting: it is the sum of the squares of the first 7 primes:

$$2^2 + 3^2 + 5^2 + 7^2 + 11^2 + 13^2 + 17^2 = 666$$

Also strange: $-2\sin(666°) = \phi = 1.618\ldots$, the golden ratio (see Fibonacci sequence section below). 666 is also the sum of the first 36 numbers, and 36 = 6^2.

IV. Sequential Wonders

Collatz's conjecture: For any number: if it's even, divide by 2. If it's odd, multiply by 3 and add 1.
Example: 45 -> 136 -> 68 -> 34 -> 17 -> 52 -> 26 -> 13 -> 40 -> 20 -> 10 -> 5 -> 16 -> 8 -> 4 -> 2 -> 1 -> 1 -> …
It seems like no matter what number you start with, you always end up with 1, 1, 1, …
No one has proven that this is true for *all* numbers, yet no counterexample has been found.

Fibonacci sequence: Start with $F_0 = 0$, $F_1 = 1$. Every number in the sequence after the first two is the sum of the preceding two numbers: $F_n = F_{n-1} + F_{n-2}$.
The sequence is therefore: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, …
Look at the ratio between elements (F_{n+1}/F_n): 1, 2, 1.5, 1.6, 1.625, 1.619, 1.618, …
The ratios converge to the *golden ratio* (denoted by the Greek letter phi): $\phi = \frac{1+\sqrt{5}}{2} = 1.618034\ldots$, which is found a lot in nature and art because of the aesthetic quality of the ratio. Interesting fact: $1/\phi = \phi - 1$.

V. Serial Wonders (advanced)

Mysterious pi: The series $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ (known as the *harmonic series*) goes to infinity, but $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \pi^2/6$, which you can prove using calculus.

The sum of integers is a fraction: $1 - 1 + 1 - 1 + 1 - \cdots = 1/2$ (*Grandi's series*)
The partial sums (the *k*th partial sum is the sum of the first *k* terms) alternate between 0 and 1. However, the *average* of the first *n* partial sums goes to 1/2 as *n* gets large: 1, 1/2, 2/3, 2/4, … (search *Cesaro sums*)
Another weird one: $1 - 2 + 3 - 4 + 5 - 6 + \cdots = 1/4$. The partial sums are 1,-1, 2, -2, 3, -3, 4, -4, … and go off to $\pm\infty$. The average of the first *n* partial sums doesn't converge either: 1, 0, 2/3, 0, 3/5, 0, 4/7, 0, …
But the average of the averages *does* go to 1/4: 1, 1/2, 5/9, 5/12, 34/75, 17/45, 298/735, 149/420, …

Taylor Series: Functions can represented by infinite series, called *Taylor series*. For example,

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad \text{and} \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

This is how many calculators compute sine, cosine, log, etc.
The development of Taylor series allowed mathematicians to prove a spectacular result:

$$e^{i\pi} + 1 = 0$$

where $i = \sqrt{-1}$ and *e* = 2.71828… This is *Euler's identity* and is one of the most beautiful math equations.