# EXERCISES FOR GEOMETRY IN MOTION

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## 1. Introduction

Here are three fun exercises to help you use the ideas seen in class on real problems.

## 2. Problems

**Problem 2.1.** This exercise will prove $\frac{d}{d\theta}\tan\theta = \frac{1}{\cos^2\theta} = \sec^2\theta$.

Draw a triangle $\triangle OPQ$ with right angle at $P$, such that $\overline{OP} = 1, \overline{PQ} = t$, and the hypotenuse $\overline{OQ} = l$, where $O$ is the origin. Draw the triangle such that $\angle QOP = \theta$. Then, we have $t = \tan\theta$. To calculate the derivative, we shift $\theta$ by a small quantity $d\theta$.

Where does the point $Q$ move? How long is $\tan(\theta + d\theta)$? Conclude from here.

**Problem 2.2** (If you know limits)**.** Use the fact that $\lim_{x\to 0}\frac{\sin x}{x} = 1$ to justify rigourously the proofs made in class.

Additionally, try to prove this limit geometrically.

**Problem 2.3.** In class, we integrated the function $x^2$ from 0 to $a$. Emulate the proof to integrate $x^n$ for all $n \geq 2$.

**Problem 2.4** (Integration and Differentiation are inverses of each other)**.** You may have heard the statement before that differentiation is the inverse of integration. We'll do so explicitly by proving that

$$\int_0^x \tan^2\theta d\theta = \tan x - x.$$

Indeed, differentiating both sides of this relation would give, as above,

$$\frac{d\tan x}{dx} - 1 = \tan^2 x \implies \frac{d\tan x}{dx} = 1 + \tan^2 x = \sec^2 x.$$

Here's the proof outline: integrate in polar coordinates the function $r(\theta) = \tan\theta$ from $\theta = 0$ to $\theta = x$. This gives you a tangent sweep. Turn it into a tangent bundle by considering the following geometric figure: draw a circle centred at $O$ of radius 1, a point $P$ on the circle, and a point $Q$ outside the circle such that $\angle POQ = x$.

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Thanks to Yale Splash for organizing everything!. You can reach me at `edgar.wang@yale.edu` if you have have questions or need hints.