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The Matrix: A Mathematical Construct

3 Perspectives on Matrices:

1. Sums

$$(\alpha+\beta)(a+b) = \begin{matrix} \alpha a + \alpha b \\ \beta a + \beta b \end{matrix}$$

Shorthand: ① drop plus signs

$$\begin{bmatrix} \alpha a & \alpha b \\ \beta a & \beta b \end{bmatrix}$$

2. Linear Transformations (maps)

$y = mx + b$ (affine)

$y = mx$ Implicit Choice

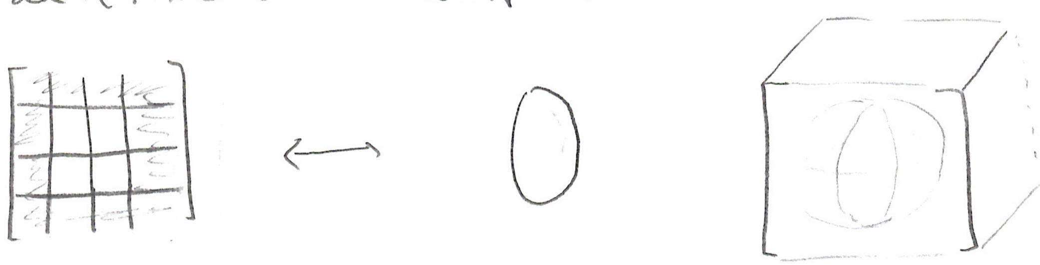
Arb Basis → N-dim

$\vec{v} = c_1 \alpha + c_2 \beta$

Shorthand: ② drop basis

$$\vec{v} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

3. Pictoral (Pixels) - comp sci



2D

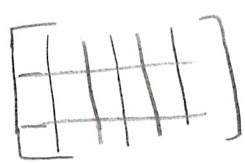
3D

Rank - # of axes

Dimension - # entries on an axis

Rank 1

Dim 3



Rank 2

Dim 3 x 2

Note: order convention in Rank 2 lost in higher dim

Motivation

$$(c_1\alpha + c_2\beta)(d_1 a + d_2 b) \xrightarrow{\text{don't know}} \begin{bmatrix} c_1 d_1 & c_1 d_2 \\ c_2 d_1 & c_2 d_2 \end{bmatrix}$$

Shorthand ① + ②

$$\xrightarrow{\text{know}} c_1 d_1 + c_2 d_2$$

∮ $a\alpha = 1$
$b\beta = 1$
$a\beta = b\alpha = 0$

How does (α, β) "play" w/ (a, b)?

$$c_1 d_1 \alpha a + c_1 d_2 \alpha b$$
$$c_2 d_1 \beta a + c_2 d_2 \beta b$$

$$M = \begin{bmatrix} \alpha a & \alpha b \\ \beta a & \beta b \end{bmatrix}$$

"Metric Tensor"

Matrix Formalism:

$$\begin{bmatrix} c_1 & c_2 \end{bmatrix} \underbrace{\begin{bmatrix} \alpha a & \alpha b \\ \beta a & \beta b \end{bmatrix}}_{M^{\mu}{}_{\nu}} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

Colm

$$\begin{bmatrix} c_1 & c_2 \end{bmatrix} \left(d_1 \begin{bmatrix} \alpha a \\ \beta a \end{bmatrix} + d_2 \begin{bmatrix} \alpha b \\ \beta b \end{bmatrix} \right) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} d_1(\alpha a) + d_2(\alpha b) \\ d_1(\beta a) + d_2(\beta b) \end{bmatrix}$$

$a \qquad b \qquad \alpha \qquad \beta$

Row Colm

$$\begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} 1^{st} \text{ row} & 1^{st} \text{ colm} \\ 2^{nd} \text{ row} & 1^{st} \text{ colm} \end{bmatrix}$$

... multiply out $c_1 d_1 \alpha a + c_1 d_2 \alpha b + c_2 d_1 \beta a + c_2 d_2 \beta b$

Hidden b/c gen work w/ $M = Id$

<u>Dual</u> - $(a, b), (\alpha, \beta)$ s.t. have well defined metric b/w them

Transformation Properties

Contravariant - opposes Δ of variable

Covariant - follows Δ of variable

Ex: "divided axis"

$10 \, \frac{m}{s} \rightarrow 1000 \, \frac{cm}{s}$ $\quad$ $10 \, m \rightarrow 1000 \, cm$

$10 \, \frac{J}{m} \rightarrow \frac{1}{10} \, \frac{J}{cm}$

Shorthand; (3) drop components

T^{μ} $\quad$ Same symbol $\Rightarrow$ dual

T_{μ}

"Einstein Notation"

dx $\quad$ contra

$\frac{d}{dx}$ $\quad$ cov

Tensor Formalism

Rnk n $\quad$ (r, s)

$\uparrow$ contra "colm" $\quad$ $\uparrow$ cov "row"

Ex:

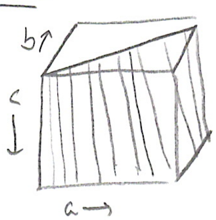
$T_{\mu}{}^{\nu}$ $\quad$ Rnk 2 $\quad$ $(1,1)$

$T_{\mu}{}^{\mu}$ $\quad$ Rnk 0 $\quad$ $(0,0)$

Tensor Contraction

$\begin{matrix}(2,3) \\ (1,0)\end{matrix} \rightarrow (2,2)$ $\quad$ contra $\cdot$ cov $\rightarrow$ #

*Diagonal metric $\Rightarrow$ trace



Why do we care?

· Physics (units) · Math (diff objects) · Less Confusion

"A tensor is what transforms like a tensor" X

Thus, "matrix"

· $\vec{v} \rightarrow \vec{v}$ $\quad$ $(1,1)$ $\quad$ map

· $\vec{v}, \vec{v} \rightarrow$ # $\quad$ $(0,2)$ $\quad$ quadratic form

· "mistake" with metric tensor $\quad$ $M_{\mu\nu}$ v $M^{\mu}{}_{\nu}$