Abstract Algebra: Questions Teachers Refused to Answer in High School

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Universal Algebra

- Set
- Operation (unary, binary, n-ary)

$\mathbb{N}$
$\mathbb{Z}$
$\mathbb{Q}$ (Q1)
$\mathbb{R}$

$\mathbb{R}[x]$ (Q2)
$\mathbb{R}^2$ ... $\mathbb{R}^3 \to$ hard
(Q3)

Group-like (1 set, 1 op)

Ex: $(\mathbb{Z}/12\mathbb{Z}, +)$ (0, inv, comm/assoc) [am v pm] abelian group

Ex: $(S_3, \circ)$ (relabel s.t. each label used once) nonabelian group

$(1\ 2\ 3)$

$(1\ 2)(2\ 3) = (1\ 2\ 3)$

$(2\ 3)(1\ 2) = (1\ 3\ 2) \equiv (3\ 2\ 1)$ inverses!

freq implicit operation

SNote: weaker cond (release... general notion)

Ex: $\mathbb{N} = \{1, 2, 3, \ldots\}$ semigroup

$\mathbb{Z}^+ = \{0, 1, 2, 3, \ldots\}$ monoid

Magma Aside

Ex: $(\mathbb{R}^3, \times)$ Cross Product

$(\hat{\imath} \times \hat{\imath}) \times \hat{\jmath} = \vec{0}$

$\hat{\imath} \times (\hat{\imath} \times \hat{\jmath}) = \hat{\imath} \times \hat{k} = -\hat{\jmath}$

Non associative

Lost All Properties!

The Rest of Group Theory:

- substructures
- construction
- classification
- unary/generalized unary op (homomorphisms)

Ring-like (1 set, 2 op)

Ex: $\mathbb{R}$, $\mathbb{Q}$ field

- caveat about zero WHY?

$0x = 0 \quad \forall x$

Pf: $\underset{\text{id \#1, op 2}}{0 \cdot x} = \underset{\text{op 1 ident}}{(0 + 0) \cdot x} = \underset{\text{dist}}{0x + 0x} \Rightarrow 0x = 0$ by inverse

See, real question is how op "play" w/ one another

Ex: $\mathbb{R}[x]$ Comm Unitary Ring

$(\mathbb{R}[x])^{\times} :=$ all units (ele w/ inverses) $= \mathbb{R}$

[Matrices are an example of a division ring (w/ ele in F)]

Q2: Solving Systems of Eqns (matrice hint)

$\frac{1}{4}x^2 + \frac{3}{4}x = -\frac{1}{2}$

$\downarrow$ $\qquad$ $\not\downarrow$

$x^2 + 3x + 2 = 0$ $\qquad$ $x^3 + 3x^2 + 2x = 0$

$(x+1)(x+2)$ $\qquad$ $x(x+1)(x+2) = 0$

$x^3 - 2x^2 = x$

$\downarrow$ $\qquad$ $\not\downarrow$

$x(x^2 - 2x - 1)$ $\qquad$ $x^2 - 2x = 1$

$x(x-1)^2$ $\qquad$ $x^2 - 2x - 1 = (x-1)^2$

* Factorization unique up to multiplication by units

$10(x-1) = 0 \equiv (x-1) = 0$ "divide by 10" (stupid thought)

Module-like

$\mathbb{R}^3$ over $\mathbb{R}$ v.s.

$\mathbb{R}^3$ over $\mathbb{Z}$ module

$M_n(\mathbb{R})$ algebra

Why just addn and mult ?

D $\quad a(b+c) = ab + ac$ $\qquad$ "Linear v Bilinear"

D' $\quad a\cdot(bc) = (a\cdot b)c = b(a\cdot c)$ ¦ $a\cdot(b+c) = a\cdot b + c = b + a\cdot c$

Ex: $7\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix} = \begin{bmatrix}7&0\\0&7\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}0&7\\0&0\end{bmatrix}$

D $\to$ "0" and "1" disjoint

D' $\to$ "0" and "1" identified gen identities

... under special circumstances

* only 2 op w/ ident distribute D (else $0 = 1$)

$\hookrightarrow$ 3 op if we distribute D' on later (not really on a single set)

Ring of Fractions

$$\frac{1}{3}+\frac{1}{4}\stackrel{?}{=}\frac{2}{7}\stackrel{?}{=}\frac{1}{12}=\frac{7}{12}$$ why this? "Criss-Cross Apple-Sauce"

Gen Desire:

- $\mathbb{Z}$ act on $\mathbb{Q}$ - classify up to units
- "real world"

$$\boxed{\frac{a}{b}=\frac{c}{d} \Leftrightarrow ad=bc}$$

This forces

$$\frac{a}{d}+\frac{c}{d}=\frac{ad+bc}{bd} \quad \text{and} \quad \frac{a}{b}\frac{c}{d}=\frac{ac}{bd} \quad \forall \text{ rings}$$

* power lies in generalization

My Notation:

set → ← operation

D 1-1	$r(v+w)=rv+rw$	linear
	$r(v\cdot w)=(rv)w$	bilinear
D' 1-2		
D 2-1	$(r+s)v=rv+sv$	linear
D' 2-2	$(rs)v=r(sv)$	bilinear

⤳ one should wonder

n-ary operation → 2^n distributive types

? release restriction

? all realised